SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2008
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #3

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators may be used.
- All necessary working should be shown in every question.
- Hand in your answers in 3 separate bundles. Section A (Question 1), Section B (Question 2) and Section C (Question 3)

Total Marks – 80

- Attempt questions 1-3
- All questions are **NOT** of equal value.

Examiner: *R. Boros*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1}, \; n \neq -1; \; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \; x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a}\sin ax, \;\; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a}\cos ax, \;\; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a}\sec ax, \;\; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a}, \;\; a > 0, \;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \;\; x > 0$

Section A – Start a new booklet.

Question 1 (27 marks). **Marks**

a) Differentiate with respect to x

(i) $\frac{x}{2}\ln\left(x^2-2\right)$ 2

(ii) $\cos^{-1}\left(3x-1\right)$ 2

b) If M and N are constants and $x = Me^{-t} + Ne^{2t}$ simplify: 4

$$\frac{d^2x}{dt^2}-\frac{dx}{dt}-2x$$

c) In how many ways can eight different books be arranged on a row such that three particular books are always together? 2

d) Find

(i) $\int \frac{1\,dx}{\left(4+x\right)^2}$

(ii) $\int \frac{1\,dx}{4+x^2}$

(iii) $\int \frac{x\,dx}{4+x^2}$ 6

e) Use the substitution $u = x-1$ to evaluate: 4

$$\int_0^1 x\left(x-1\right)^5 dx$$

f) What is the largest possible domain of: 3

$$y = \ln\left(x^2-1\right)$$

g) Four digit numbers greater than 7000 are to be made from the digits 8, 7, 6 and 5. Repetition of the digits is allowed. How many such numbers can be made? 2

h) (i) How many different arrangements of the letters of BALLOON are possible? 1

(ii) How many of these start with the letter L. 1

End of Section A.

Section B – Start a new booklet.

Question 2 (29 Marks). **Marks**

a) Consider the function $f(x) = 2\tan^{-1}\left(\frac{x}{2}\right)$.

(i) Evaluate $f(0)$. 1

(ii) Draw the graph of $f(x)$. 1

(iii) State the domain and range. 2

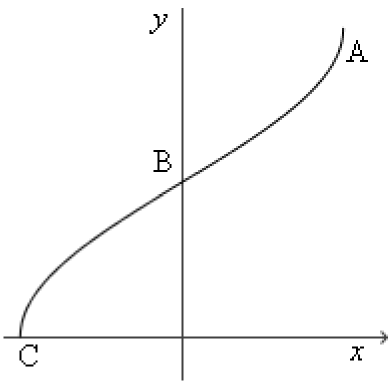
b) For the function $y = x + e^{-x}$

(i) Find the coordinates and determine the nature of any stationary points on the graph of $y = f(x)$. 3

(ii) Show that the graph is always concave upwards for all values of x. 1

(iii) Sketch the graph of $y = f(x)$ showing clearly the coordinates of any stationary points and the equations of any asymptotes. 2

c) The diagram below shows the graph of $y = \pi + 2\sin^{-1} 3x$.



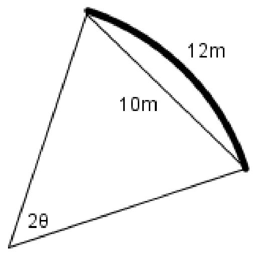
(i) Write down the coordinates of the point B. 2

(ii) Write down the coordinates of the endpoints A and C. 4

(iii) Find the equation of the tangent to the curve $y = \pi + 2\sin^{-1} 3x$ at the point B (give answer in general form of equation) 3

Section B continues next page

d) A pipe which is 12 metres long is bent into a circular arc which subtends an angle of 2θ radians at the centre of the circle. The chord of the circle joining the ends of the arc is 10 metres long.



(i) Show that $6\sin\theta - 5\theta = 0$. 3

(ii) Show that $\theta_0 = 1$ is a good first approximation to the value of θ. 2

(iii) Use one application of Newton's Method to find another approximation, θ_1, to the value of θ (correct to 4 decimal places). 3

(iv) Use this value of θ_1, to find an approximation to the length of the radius of the arc, rounding off this answer correct to 2 decimal places. 2

End of Section B.

Section C – Start a new booklet.

Question 3 (24 marks). **Marks**

a)

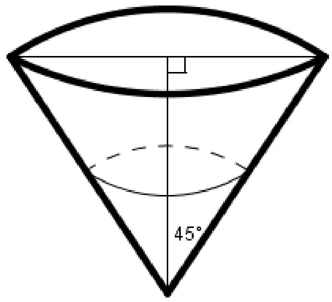


Figure not to scale.

Water is being let into the conical vessel, shown above, at a constant rate of 8cm^3/second.

(i) Show that if *h* cm is the depth of the vessel then $r = h$ and

$V = \frac{1}{3}\pi h^3$ 2

When the depth is 12cm find:

(ii) The rate of increase in the depth (in terms of π). 3

(iii) The rate of increase in the area of the top surface of the water. 3

b) The rate of decrease of temperature of a body hotter than its surroundings is proportional to the temperature difference. If *A* is the air temperature and *T* is the temperature of the body after *t* minutes, then

$$\frac{dT}{dt} = -k(T - A) \qquad (1)$$

(i) Show that if *I* is the initial temperature of the body, then the function $T = A + \left(I - A\right)e^{-kt}$ satisfies the condition (1). 2

(ii) IF the initial temperature of an ingot is $1400^o C$ and it cools in the open where the air temperature is $20^o C$, find the temperature after 30 minutes, given that it cooled to $1200^o C$ in 5 minutes. 4

Section C continues next page

c) The acceleration of a particle moving in a straight line is given by

$$\ddot{x} = -\frac{900}{x^3}$$

Where x metres is the displacement from the origin after t seconds. Initially, the particle is 10 metres to the right of the origin, moving with a velocity of 3m/s.

(i) Find an equation for the velocity of the particle. 3

(ii) Find the velocity of the particle when it is 100m to the right of the origin. 1

d) The curves $y = \sin^{-1} x$ and $y = \cos^{-1} x$ intersect at point P. The acute angle between their tangents at that point is θ. Find θ to the nearest degree. You may need to use the fact that $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$, where m_1 and m_2 are the gradients of the tangent lines. 6

End of Section C.

End of Examination.

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Solutions to Section (A)

(1)

(a) $\frac{d}{dx}\left[\frac{x}{2}\ln(x^2-2)\right]$

(i) $= \frac{1}{2}\ln(x^2-2) + \frac{x^2}{x^2-2}$ **2**

or $1 + \frac{1}{2}\ln(x^2-2) + \frac{2}{x^2-2}$.

(ii) $\frac{d}{dx}\cos^{-1}(3x-1)$

$= \frac{-3}{\sqrt{1-(3x-1)^2}}$ **2**

OR $\frac{-3}{\sqrt{3(2x-3x^2)}}$

(b) $x = Me^{-t} + Ne^{2t}$

$\dot{x} = -Me^{-t} + 2Ne^{2t}$

• $\ddot{x} = Me^{-t} + 4Ne^{2t}$.

4

$\therefore \ddot{x} - \dot{x} - 2x$

$= Me^{-t} + 4Ne^{2t} + Me^{-t} - 2Ne^{2t}$

• $- 2Me^{-t} - 2Ne^{2t}$

$= 0$

(c) $\boxed{}$ - - - - -

$6! \times 3!$ **2**

(d) (i) • $\int (4+x)^{-2}\,dx = \frac{-1}{4+x} + c$

(ii) • $\frac{1}{2}\tan^{-1}\frac{x}{2} + c$

(iii) • $\frac{1}{2}\int \frac{2x}{4+x^2}\,dx = \frac{\ln(4+x^2)}{2} + c$.

6

(e) $\int_0^1 x(x-1)^5\,dx$

Let $u = x-1$, $du = dx$

• $x=0, u=-1$. $x=1, u=0$.

4

$\int_{-1}^{0}(1+u)u^5\,du$

$= \int_{-1}^{0}(u^5+u^6)\,du$

• $= \left[\frac{u^6}{6}+\frac{u^7}{7}\right]_{-1}^{0}$

$= 0 - \left(\frac{1}{6}-\frac{1}{7}\right) = -\frac{1}{42}$.

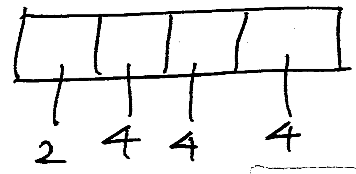
(f) $x^2 - 1 > 0$

$\Rightarrow$ • $x^2 > 1$ **3**

either $x > 1$ •

or $x < -1$ •

(g) > 7000 (repetition)



$= 2 \times 4^3 = 128$. **2**

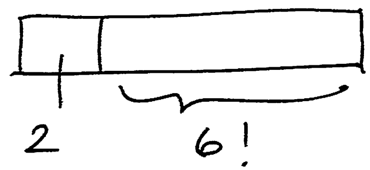
(h)

(i) BALLOON

(7 letters).

(i) $\frac{7!}{2!\,2!} = 1260$ **1**

(ii)



$= 2 \times \frac{6!}{2!\,2!}$

$= \frac{1}{2}(6!) = 360$.

1

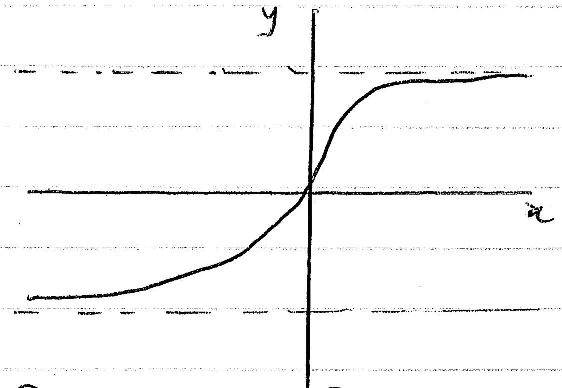
[27]

Ext1 TASK 3

Section B.

a) i) $f(0)=0$.

ii)



iii) Domain: $x \in \mathbb{R}$.

Range: $-\pi < y \leq \pi$.

b). i) $y = x + e^{-x}$

$\frac{dy}{dx} = 1 - e^{-x}$

$1 - e^{-x} = 0$

$x = 0$.

$y = 1$.

$\frac{d^2y}{dx^2} = e^{-x}$

At $(0,1)$.

$e^0 > 0$

$\therefore$ $(0,1)$ is a local minimum.

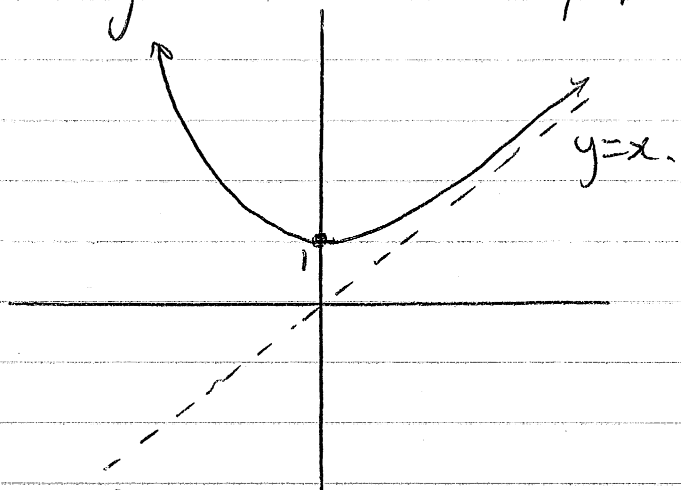
ii) $\frac{d^2y}{dx^2} = e^{-x} > 0 \quad \forall x$.

Therefore the graph is always concave up.

iii)

$$\lim_{x\to\infty}[y-x] = \lim_{x\to\infty} e^{-x}$$
$$= 0.$$

$\therefore$ $y=x$ is an asymptotes.



c i). $B(0,\pi)$.

ii) $A(\frac{1}{3}, 2\pi)$
$C(-\frac{1}{3}, 0)$.

iii). $y = \pi + 2\sin^{-1}3x$

$$\frac{dy}{dx} = 2\times\frac{1}{\sqrt{1-(3x)^2}}\times 3.$$

$$= \frac{6}{\sqrt{1-9x^3}}$$

At $x=0$.

$m=6$

$y-y_1 = m(x-x_1)$.

$y-\pi = 6x$.

$6x - y + \pi = 0$.

d) i)

$$\frac{r}{\sin\left(\frac{\pi-2\theta}{2}\right)} = \frac{10}{\sin 2\theta}.$$

$$l = r\theta.$$

$$12 = r\,2\theta$$

$$r = \frac{6}{\theta}.$$

$$\frac{r}{\cos\theta} = \frac{10}{2\sin\theta\cos\theta}$$

$$r = \frac{5}{\sin\theta}$$

$$\therefore \frac{5}{\sin\theta} = \frac{6}{\theta}.$$

$$5\theta = 6\sin\theta.$$

$$6\sin\theta - 5\theta = 0.$$

ii). Let $f(\theta) = 6\sin\theta - 5\theta$

$$f(1) = 6\sin(1) - 5\times 1,$$

$$\approx 0.0488$$ which is close to zero.

iii) $\theta_1 = \theta_0 - \frac{f(\theta_0)}{f'(\theta_0)}$

$$f'(\theta) = 6\cos\theta - 5.$$

$$\theta_1 = 1 - \frac{f(1)}{f'(1)}$$

$$\approx 1.0278.$$

iv) $r = \frac{6}{\theta_1}$

$$\approx 5.84.$$

2008 Assessment #3 Mathematics Extension 1:
Solutions— Section C

3. (a)

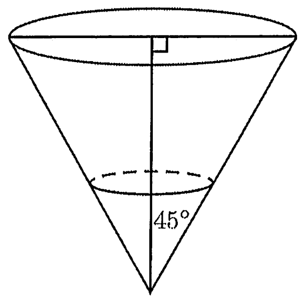
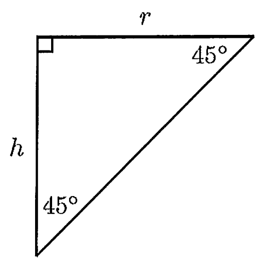


Figure not to scale.
Water is being let into the conical vessel, shown above, at a constant rate of $8\,\text{cm}^3/\text{second}$.

(i) Show that if $h\,\text{cm}$ is the depth of the vessel then $r = h$ and $V = \frac{1}{3}\pi h^3$. **2**

Solution:



$$\begin{aligned}\tan 45° &= \frac{r}{h},\\ &= 1.\\ \therefore\ r &= h.\end{aligned}$$

{OR: As the triangle is isosceles, (base angles equal) then $r = h$.}

Volume of cone, $$\begin{aligned}V &= \frac{1}{3}\pi r^2 h,\\ &= \frac{1}{3}\pi h^3.\end{aligned}$$

When the depth is 12 cm find:

(ii) The rate of increase in the depth (in terms of π). **3**

Solution:

$$\begin{aligned}\frac{dV}{dt} &= 8\,\text{cm}^3\text{s}^{-1},\\ \frac{dV}{dh} &= \pi h^2,\\ \frac{dh}{dt} &= \frac{dV}{dt} \times \frac{dh}{dV},\\ &= \frac{8}{\pi h^2},\\ &= \frac{1}{18\pi}\,\text{cm s}^{-1} \text{ when } h = 12\,\text{cm}.\end{aligned}$$

(iii) The rate of increase in the area of the top surface of the water. [3]

> **Solution:** Surface area, $S = \pi h^2$,
> $$\frac{dS}{dh} = 2\pi h$$
> $$\begin{aligned} \therefore \quad \frac{dS}{dt} &= \frac{dS}{dh} \times \frac{dh}{dt}, \\ &= 2\pi h \times \frac{8}{\pi h^2}, \\ &= \frac{16}{h}, \\ &= \frac{4}{3}\,\text{cm}^2\text{s}^{-1} \text{ when } h = 12\,\text{cm}. \end{aligned}$$

(b) The rate of decrease of temperature of a body hotter than its surroundings is proportional to the temperature difference. If A is the air temperature and T is the temperature of the body after t minutes, then

$$\frac{dT}{dt} = -k(T - A) \qquad (1)$$

(i) Show that if I is the initial temperature, then the function $T = A + (I - A)e^{-kt}$ satisfies the condition (1). [2]

> **Solution:**
> $$\begin{aligned} \frac{dT}{dt} &= -k(I - A)e^{-kt}, \\ \text{but } (I - A)e^{-kt} &= T - A, \\ \therefore \quad \frac{dT}{dt} &= -k(T - A). \end{aligned}$$

(ii) If the initial temperature of an ingot is 1400°C and it cools in the open where the air temperature is 20°C, find the temperature after 30 minutes, given that it cooled to 1200°C in 5 minutes. [4]

> **Solution:**
> $$\begin{aligned} 1200 &= 20 + (1400 - 20)e^{-5k}, \\ e^{-5k} &= \frac{1180}{1380}, \\ -5k &= \ln\left(\frac{59}{69}\right), \\ k &= \frac{\ln\left(\frac{59}{69}\right)}{-5}, \\ &\approx 0.0313. \\ T_{30} &\approx 20 + (1400 - 20)e^{-30 \times 0.0313}, \\ &\approx 559.4°C \text{ (4 sig. fig.)}. \end{aligned}$$

(c) The acceleration of a particle moving in a straight line is given by

$$\ddot{x} = -\frac{900}{x^3}$$

where x metres is the displacement from the origin after t seconds. Initially, the particle is 10 metres to the right of the origin, moving with a velocity of 3 m/s.

(i) Find an equation for the velocity of the particle. [3]

Solution:

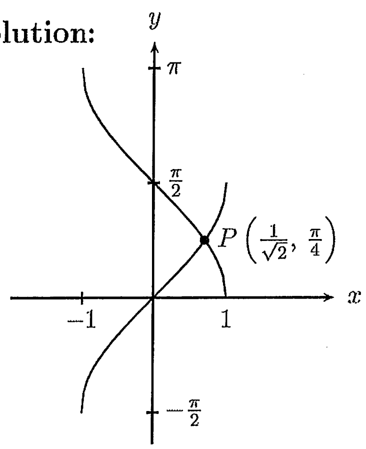
$$\begin{aligned} v\frac{dv}{dx} &= -\frac{900}{x^3}, \\ \int v\,dv &= -900\int x^{-3}\,dx, \\ \frac{v^2}{2} &= \frac{900x^{-2}}{2} + c, \\ i.e.\ v^2 &= \frac{900}{x^2} + c. \\ \text{Initially } 9 &= \frac{900}{100} + c; \\ \therefore\ v &= \frac{30}{x} \quad \text{(positive velocity from initial conditions).} \end{aligned}$$

(ii) Find the velocity of the particle when it is 100 m to the right of the origin. [1]

Solution: When $x = 100$ m, $v = {}^3\!/_{10}\,\text{ms}^{-1}$.

(d) The curves $y = \sin^{-1} x$ and $y = \cos^{-1} x$ intersect at point P. The acute angle between their tangents at this point is θ. Find θ to the nearest degree. You may need to use the fact that $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$, where m_1 and m_2 are the gradients of the tangent lines. [6]

Solution:



From the sketch it is clear that P is $\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$.

$$\begin{aligned} \frac{d\,(\sin^{-1} x)}{dx} &= \frac{1}{\sqrt{1-x^2}}, \\ &= \sqrt{2} \text{ at } P. \\ \frac{d\,(\cos^{-1} x)}{dx} &= \frac{-1}{\sqrt{1-x^2}}, \\ &= -\sqrt{2} \text{ at } P. \\ \tan\theta &= \left|\frac{\sqrt{2} - {}^-\sqrt{2}}{1 + \sqrt{2} \times ({}^-\sqrt{2})}\right|, \\ &= \left|\frac{2\sqrt{2}}{-1}\right|. \\ \therefore\ \theta &= \tan^{-1}(2\sqrt{2}), \\ &\approx 71^\circ \text{ (nearest degree).} \end{aligned}$$